RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

**B.A./B.Sc. SIXTH SEMESTER EXAMINATION, JULY 2021**

**THIRD YEAR [BATCH 2018-21]**

Date : 10/07/2021 **MATHEMATICS**

Time : 11.00 am – 3.00 pm **Paper** : MTM P 7 Full Marks:100

## Instructions to the students

- Write your **College Roll No, Year, Subject** & **Paper Number** on the top of the **Answer Script**.
- Write your **Name, College Roll No, Year, Subject** & **Paper Number** on the **text box of your e-mail**.
- Read the instructions given at the beginning of each paper/group/unit carefully.
- Only handwritten (by blue/black pen) answer-scripts will be permitted.
- Try to answer all the questions of a single group/unit at the same place.
- All the pages of your answer script must be numbered serially by hand.
- In the last page of your answer-script, please mention the total number of pages written so that we can verify it with that of the scanned copy of the script sent by you.
- For an easy scanning of the answer script and also for getting better image, students are advised to write the answers in single side and they must give a minimum 1 inch margin at the left side of each paper.
- After the completion of the exam, scan the entire answer script by using Clear Scan: Indy Mobile App OR any other Scanner device and make a **single PDF file (Named as your College Roll No)** and send it to [redacted]

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**Group - A** (Analysis IIIB)

Answer all the questions, maximum you can score is 30.

1. Find the Fourier series of $f(x)$ with period $2\pi$, where $f(x) = 0$, for $-\pi < x < a$; $f(x) = 1$ for $a \le x \le b$; $f(x) = 0$ for $b < x < \pi$. Find the sum of the series for $x = 4\pi + a$ and deduce that $\sum\limits_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi-b+a}{2}$. [5+2+1]

2. Verify Gauss' theorem for $\iint_S (2x - z)dydz + x^2ydzdx - z^2xdxdy$ taken over the surface $S$ bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. [8]

3. Prove that $\int_0^{\infty} \frac{x^m(1+x^n)}{1+x^p}dx$ $(m > 0, n > 0)$ is convergent if $p > 1 + m + n$. [4]

4. Obtain the Fourier series expansion of $f(x) = x\sin x$ on $[-\pi, \pi]$. Hence deduce that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots$ [5+1]

5. Verify Green's theorem in the plane for $\int_\Gamma (3x^2-8y^2)dx+(4y-6xy)dy$, where $\Gamma$ is the boundary of the region defined by $y=\sqrt{x}, y=x^2$. [6]

6. Prove that $\int_0^\infty x^m(\log x)^n dx$ is convergent if $m<-1$. [4]

## Group B (Number Theory)

Answer all the questions, maximum you can score is 20.

7. (a) If $a$ is even, prove that $48 \mid a(a^2+20)$.

   (b) If $n \in \mathbb{N}$, show that $\dfrac{15n+4}{12n+3}$ is not an integer.

   (c) Prove that the only prime $p$ such that $p+1$ is a perfect square is 3.

   (d) If $a$ is prime to $b$, prove that $a^2+b^2$ is prime to $a^2b^2$. [4+2+2+5=13]

8. The sum of two positive integers is 200. If one is divided by 5 and the other is divided by 9, the remainder is 1 in each case. Find the two numbers. [5]

9. (a) Suppose $n=10a+b$, where a, b are integers. Prove that $n$ is divisible by 7 if $a-2b$ is divisible by 7.

   (b) Use the theory of congruences to prove that $7 \mid 2^{5n+3}+5^{2n+3}$, for all $n \in \mathbb{N}$. [3+3=6]

## Group C (Probability and Statistics)

Answer any **three** questions from Q10 to Q13.

[3 x 10=30]

10. (a) Two dice are thrown n times in succession. What is the probability of obtaining 'double six' at least once? Also find the least number of throws so that the probability of obtaining at least one 'double six' will be greater than $\frac{1}{2}$. [3+3]

    (b) Consider all families with two children and assume that each child is equally likely to be a boy or a girl. If such a family is picked up at random and found to have a boy, then what is the probability that it has another boy? [4]

11. (a) Three concentric circles of radii $\dfrac{1}{\sqrt{3}}$, 1, $\sqrt{3}$ are drawn on a target board. If a shot falls within the innermost circle 3 points are scored; if it falls within the next two rings, the scores are respectively 2 and 1; and the score is 0 if the shot is outside the outermost circle. If the pdf of the distance (r) of the hit from the centre of the target is $\dfrac{2}{\pi}.\dfrac{1}{1+r^2}$, find the probability distribution of the score. [5]

    (b) The joint probability density function of two random variables X and Y is

$$f(x,y)=\begin{cases} 8xy, 0\le x\le y, 0\le y\le 1 \\ 0, \text{ otherwise.} \end{cases}$$

    Examine whether X and Y are independent. Also compute Var(X) and Var(Y). [3+1+1]

12. (a) Find the mean and the median of the distribution given by the probability density function

$$f_x(x)=kx(1-x),\ 0\le x\le 1,$$

    where k is a suitable constant to be calculated. [3+3]

(b) Find the correlation coefficient between the random variables 2X-3 and X+2. [4]

13. (a) The joint probability density function of the random variables X and Y is

$$f(x,y) = \begin{cases} k(1-x-y), x \geq 0, y \geq 0, x+y \leq 1 \\ 0, \text{ elsewhere,} \end{cases}$$

where k is a constant. Find (i) the value of k, (ii) the covariance of X and Y. [2+4]

(b) Show by Tchebycheff's inequality that in 2000 throws with an unbiased coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$. [4]

## Group D (Complex Analysis)

Answer all the questions, maximum one can score 20.

14. If $\lim_{z \to z_0} f(z) = w_0$, show that $\lim_{z \to z_0} \overline{f(z)} = \overline{w_0}$. [2.5]

15. Find the domain of convergence of the following series and give the geometric nature of the obtained domain:

(a) $\sum_{n=1}^{\infty} \left( \frac{2i}{z+1+i} \right)^n$ [2.5]

(b) $\sum_{n=1}^{\infty} \left( \frac{1.3.5...(2n-1)}{n!} \right) \left( \frac{1-z}{z} \right)^n$ [3.5]

16. Let $f$ be an entire function. If $f(z) = 0 \; \forall \; z \in D$ where $D = \left\{ 5e^{it} : \; t \in \left[0, \frac{\pi}{4}\right] \right\}$ then find $f(0)$. [4]

17. If $f(z)$ is a non-vanishing analytic function in some domain D, show that $\ln |f(z)|$ is harmonic in D. [5]

18. Construct an analytic function $f(z) = u(x,y) + iv(x,y)$ for which real part is given by $u(x,y) = (e^x + \sinh x) \cos y - 4y$. [3.5]

19. Solve: $\sinh(5z+i) = -\sqrt{3}\, i$ [3]

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